Announcements

1) Extra credit to be posted on CTools sometime later tonight

Due Thursday after break

2) Quiz Thursday after break
11.1, 11.2, 11.3

Where's the Calculus?

Definition: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.

The sequence is said to converge to a number L if for every $\varepsilon > 0$, there is a counting number N with

$$|a_n - L| < \varepsilon$$

for all $n \geq N$.

The number L is called the limit of the sequence – we'll never use this definition!

Use the following result:

If there is a function f defined $[1, \infty)$ satisfying

$$a_n = f(n)$$

for all counting numbers n, then the limit of $(a_n)_{n=1}^{\infty}$ is equal to L if

$$\boxed{\lim_{x \to \infty} f(x) = L}$$

Example 1: Find $\lim_{n\to\infty} \frac{7n^2+2n}{6n^2+7}$.

Find $\lim_{x\to\infty} \frac{7x^2+2x}{6x^2+7}$ - If the limit exists, then that's the limit of the sequence.

Either by looking at highest powers $\left(\lim_{x\to\infty} \frac{7x^2+2x}{6x^2+7} = \lim_{x\to\infty} \frac{7x^2}{6x^2} = \frac{7}{6} \right)$

or by using l'Hopital's rule,

we get $\lim\limits_{x \to \infty} \dfrac{7x^2+2x}{6x^2+7} = \dfrac{7}{6}$,

so $\lim\limits_{n \to \infty} \dfrac{7n^2+2n}{6n^2+7} = \boxed{\dfrac{7}{6}}$

Example 2: $\lim_{n \to \infty} \frac{n^2}{2^n}$.

$$\lim_{x \to \infty} \frac{x^2}{2^x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2x}{\ln(2)\, 2^x}$$

$$\overset{L'H}{=} \lim_{x \to \infty} \frac{2}{(\ln(2))^2\, 2^x}$$

constant

$\to \infty$

$$= 0$$

So $\lim_{n \to \infty} \frac{n^2}{2^n} = 0$.

Some familiar theorems come back:

Squeeze Theorem for Sequences

If $a_n \le b_n \le c_n$ for all counting numbers n,

then if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$,

we have $\lim_{n\to\infty} b_n = L$

Example 3: Find $\lim_{n \to \infty} \frac{\arctan(n)}{\sqrt{n}}$.

We know

$$-\frac{\pi}{2} \le \arctan(n) \le \frac{\pi}{2}$$

Dividing across by $\sqrt{n}$,

$$\frac{\left(-\frac{\pi}{2}\right)}{\sqrt{n}} \le \frac{\arctan(n)}{\sqrt{n}} \le \frac{\left(\frac{\pi}{2}\right)}{\sqrt{n}}$$

take limits.

$$\underbrace{\lim_{n \to \infty} \frac{\left(-\frac{\pi}{2}\right)}{\sqrt{n}}}_{=0} \le \lim_{n \to \infty} \frac{\arctan(n)}{\sqrt{n}} \le \underbrace{\lim_{n \to \infty} \frac{\left(\frac{\pi}{2}\right)}{\sqrt{n}}}_{=0}$$

So by the squeeze theorem,

$$\lim_{n\to\infty} \frac{\arctan(n)}{\sqrt{n}} = \boxed{0}$$

Example 4:

$$\lim_{n \to \infty} r^n \quad \text{for a real number } r$$

$r = 12$

$$\lim_{n \to \infty} (12)^n = \infty$$

$r = .5 = \frac{1}{2}$

$$\lim_{n \to \infty} \left(\frac{1}{2}\right)^n = \lim_{n \to \infty} \frac{1^n}{2^n}$$

$$= \lim_{n \to \infty} \frac{1}{2^n} = 0$$

$r = -2$

$((-2)^n) = (-2, 4, -8, 16, -32, \ldots)$

Since the limit of the odd terms is $-\infty$ and the limit of the evens is ∞,

$\lim_{n \to \infty} (-2)^n$ does not exist

$r = 1$, $1^n = 1$ for all n,

so $\lim_{n \to \infty} 1^n = \lim_{n \to \infty} 1 = 1$.

$$\lim_{n\to\infty} r^n = \begin{cases} \infty, & \text{if } r>1 \\ 1, & \text{if } r=1 \\ 0, & \text{if } -1<r<1 \\ \text{does not exist}, & \text{if } r\leq -1 \end{cases}$$

Note: $(-1)^n = (-1, 1, -1, 1, -1, 1, \_\_\_)$

odd terms are negative one,
even terms are one, so
no limit!

Example 5: $\lim_{n\to\infty} \frac{(23)^n}{n!}$

$$n! = n(n-1)(n-2)\cdot(n-3) — \cdot 2 \cdot 1$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 5040$$

Note: $n! = n\Big(\underbrace{(n-1)\cdot(n-2)\cdot — \cdot 2 \cdot 1}\Big)$

$$= n\big((n-1)!\big)$$

Save limit for later - its zero ...

Infinite Series

(Section 11.2)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{---}$$

we said this was 1

We just figured out that

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \text{---}$$

we said this was $\frac{1}{2}$

Let r be a real number

I want a formula for

$$r + r^2 + r^3 + \cdots + r^{n-1} + r^n = S_n.$$

$$S_{n+1} = \underbrace{r + r^2 + r^3 + \cdots + r^{n-1} + r^n}_{\parallel} + r^{n+1}$$

$$= S_n + r^{n+1}$$

Also, we can factor out an r

$$S_{n+1} = r\left(1 + r + r^2 + \cdots + r^{n-1} + r^n\right)$$

$$= r\left(1 + S_n\right)$$

Then

$$S_{n+1} = S_n + r^{n+1} = r(1 + S_n)$$

so

$$S_n + r^{n+1} = r + rS_n$$

$$S_n - rS_n = r - r^{n+1}$$

factor out S_n

$$S_n(1-r) = r - r^{n+1}$$

If $r \neq 1$, divide both sides by $(1-r)$

$$S_n = \frac{r - r^{n+1}}{1-r}$$

If $r = 1$

$$S_n = 1 + 1^2 + 1^3 + 1^4 + \cdots + 1^{n-1} + 1^n$$

$$= \underbrace{1 + 1 + 1 + 1 + \cdots + 1 + 1}_{n \text{ ones}}$$

$$= n$$

What happens when we take

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r - r^{n+1}}{1-r} \quad (r \neq 1)$$

$$= \frac{r - \lim_{n\to\infty} r^{n+1}}{1-r}$$

$$\lim_{n\to\infty} r^{n+1} = \begin{cases} 0, & -1 < r < 1 \\ \infty, & r > 1 \\ \text{does not exist}, & r \leq -1 \end{cases}$$

If $r > 1$ or $r \leq -1$,

$\lim_{n \to \infty} S_n$ is not a real number.

If $-1 < r < 1$,

$$\lim_{n \to \infty} S_n = \frac{r - \lim_{n \to \infty} r^{n+1}}{1 - r} \quad = 0$$

$$= \frac{r}{1-r}$$

Yours to use for the rest of class!

If $r = 1$,

$S_n = n$, so

$$\lim_{n \to \infty} S_n = \infty.$$

Check with $r = \frac{1}{2}$:

$$\lim_{n \to \infty} S_n = \frac{r}{1-r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$= 1$$

Generalization: what if

$(a_k)_{k=1}^{\infty}$ is some sequence

and $a_k \neq r^k$?

Can we still figure out

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (a_1 + a_2 + a_3 + \cdots + a_n) \, ?$$

Idea: write

$$a_1 + a_2 + a_3 + \text{---} + a_n$$

$$= \sum_{k=1}^{n} a_k = S_n.$$

If $\lim_{n \to \infty} S_n$ exists, we say the sum converges

and we write $\sum_{k=1}^{\infty} a_k$ for the limit

If $\lim_{n\to\infty} S_n$ does not exist

(is not a real number)

we say the series

diverges.

In this class, a sequence is a list of real numbers $(a_n)_{n=1}^{\infty}$. A series is what you get when you add up the terms of a sequence.

Geometric Series

$\boxed{\sum_{n=1}^{\infty} r^n}$ for r a real number.

Converges for $-1 < r < 1$ to $\frac{r}{1-r}$

Diverges for $|r| \geq 1$.

If a is any real number and $-1 < r < 1$,

$$\boxed{\sum_{n=1}^{\infty} ar^n = a\left(\frac{r}{1-r}\right)}$$

yours to use, without justification.

But use it correctly!

Example 6: Find

$$\sum_{n=2}^{\infty} \frac{7^{2n+1}}{123^{n+5}} = ?$$

Want to get to the point where you can use the formula

$$\sum_{n=1}^{\infty} ar^n = a\left(\frac{r}{1-r}\right)$$

Will take work!

Look at $\dfrac{7^{2n+1}}{123^{n+5}}$

$$= \frac{7^{2n} \cdot 7}{123^{n}\, 123^{5}}$$

some number, no n's

$$= \frac{7}{123^5} \cdot \frac{7^{2n}}{123^n}$$

$$= \frac{7}{123^5} \frac{(7^2)^n}{123^n}$$

$= r$

$$= \frac{7}{123^5} \left(\frac{49}{123}\right)^n$$

We then get

$$\sum_{n=2}^{\infty} \frac{7^{2n+1}}{123^{n+5}} = \sum_{n=2}^{\infty} \frac{7}{123^5}\left(\frac{49}{123}\right)^n$$

$$= \sum_{n=2}^{\infty} \frac{7}{123^5}\left(\frac{49}{123}\right)\left(\frac{49}{123}\right)^{n-1}$$

Setting $m = n-1$, we get

$(n=2,\ m=2-1=1)$

$$= \sum_{m=1}^{\infty} \underbrace{\frac{7}{(123)^5}\left(\frac{49}{123}\right)}_{a}\left(\frac{49}{123}\right)^m$$

$|r|<1$

Use formula

The series converge to

$$a\left(\frac{r}{1-r}\right) = \boxed{\frac{7}{123^5}\cdot\frac{49}{123}\cdot\left(\frac{49/123}{1-49/123}\right)}$$